

# On projective invariants of the complex Finsler spaces

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## Abstract

In this paper the projective curvature invariants of a complex Finsler space are obtained. By means of these invariants the notion of complex Douglas space is then defined. A special approach is devoted to obtain the equivalence conditions that a complex Finsler space should be Douglas. It is shown that any weakly Kähler Douglas space is a complex Berwald space. A projective curvature invariant of Weyl type characterizes the complex Berwald spaces. They must be either purely Hermitian of constant holomorphic curvature or non purely Hermitian of vanish holomorphic curvature. The locally projectively flat complex Finsler metrics are also studied.

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## 1 Introduction

The study of projective real Finsler spaces was initiated by L. Berwald, [10, 11], and his studies mainly concern the two dimensional Finsler spaces. Further substantial contributions on this topic came later from Rapcsák [22], Misra [19] and, especially, from Z. Szabo [27] and M. Matsumoto [17]. The problem of projective Finsler spaces is strongly connected to projectively related sprays, as Z. Shen pointed out in [26]. The topic of projective real Finsler spaces continues to be of interest for some projective invariants: Douglas curvature, Weyl curvature and others. The exploration of these projective invariants leads to the special classes of metrics such as the Douglas metrics and the Finsler metrics of scalar flag curvature, ([7, 8, 14, 16, 12], etc.).

Few general themes from projective real Finsler geometry are broached in complex Finsler geometry, [3]. Two complex Finsler metrics  $L$  and  $\tilde{L}$ , on a common underlying manifold  $M$ , are called projectively related if any complex geodesic curve, in [1]'s sense, of the first is also complex geodesic curve for the second and the other way around. This means that between the spray coefficients  $G^i$  and  $\tilde{G}^i$  there is a so called projective change  $\tilde{G}^i = G^i + B^i + P\eta^i$ , where  $P$  is a smooth function on  $T'M$  with complex values and  $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$ . Although the Chern-Finsler complex nonlinear connection, with the local coefficients  $N_j^i$ , is the main tool in complex Finsler geometry ([1, 20]), in this study we use the canonical complex nonlinear connection because it derives from a complex spray, i.e.  $N_j^i := \dot{\partial}_j G^i$  and  $G^i = \frac{1}{2}N_j^i\eta^j$ .

Using some ideas from the real case, our aim in the present paper is to study the above mentioned projective change. It gives rise to projective curvature invariants of Douglas and Weyl types. Associated to the canonical complex nonlinear connection we have the complex linear connection of Berwald type which is an important tool in our approach.

Subsequently, we have made an overview of the paper's content.

In §2, some preliminary properties of the  $n$  - dimensional complex Finsler spaces are stated. We prove that the complex Finsler spaces which are weakly Kähler and generalized Berwald are complex Berwald spaces (Theorem 2.1).

In §3, the structure equations satisfied by the connection form of the complex linear connection of Berwald type are emphasized. Next, we derive some of Bianchi identities which specify the relations among the covariant derivatives of the curvature coefficients of this complex linear connection.

A first class of projective curvature invariants obtained by successive vertical differentiations of the projective change is explored in §4. We find three projective invariants of Douglas type and by means of them are defined the complex Douglas spaces. The necessary and sufficient conditions in which a complex Finsler space is Douglas are contained in Theorem 4.2.

The study of the weakly Kähler projective changes is more significant. We prove that the weakly Kähler Douglas spaces are complex Berwald spaces, (Theorem 5.2). A projective curvature invariant of Weyl type  $W_{jkh}^i$ , which has the same formal form as in the real case, is obtained. It is vanishing in the Kähler context. For the complex Berwald spaces another projective curvature invariant of Weyl type  $W_{j\bar{k}h}^i$  is found. We show that  $W_{j\bar{k}h}^i = 0$  if and only if the space is either purely Hermitian with the holomorphic curvature  $\mathcal{K}_F$  equal to a constant value or non purely Hermitian with  $\mathcal{K}_F = 0$ , (Theorem 5.4).

The last part of the paper, §6, is devoted to the locally projectively flat complex Finsler metrics. The necessary and sufficient conditions for the

locally projectively flat complex Finsler metrics and other characterizations are established in Theorems 6.2, 6.3 and Proposition 6.2. Finally, the locally projectively flat complex Finsler metrics are exemplified, better illustrating the interest for this work, (Theorem 6.4).

## 2 Preliminaries

Let  $M$  be a  $n$ -dimensional complex manifold and  $z = (z^k)_{k=\overline{1,n}}$  be the complex coordinates in a local chart. The complexified of the real tangent bundle  $T_C M$  splits into the sum of holomorphic tangent bundle  $T'M$  and its conjugate  $T''M$ . The bundle  $T'M$  is itself a complex manifold and the local coordinates in a local chart will be denoted by  $u = (z^k, \eta^k)_{k=\overline{1,n}}$ . These are changed into  $(z'^k, \eta'^k)_{k=\overline{1,n}}$  by the rules  $z'^k = z'^k(z)$  and  $\eta'^k = \frac{\partial z'^k}{\partial z^l} \eta^l$ .

A *complex Finsler space* is a pair  $(M, F)$ , where  $F : T'M \rightarrow \mathbb{R}^+$  is a continuous function satisfying the conditions:

- i)  $L := F^2$  is smooth on  $\widetilde{T'M} := T'M \setminus \{0\}$ ;
- ii)  $F(z, \eta) \geq 0$ , the equality holds if and only if  $\eta = 0$ ;
- iii)  $F(z, \lambda\eta) = |\lambda|F(z, \eta)$  for  $\forall \lambda \in \mathbb{C}$ ;
- iv) the Hermitian matrix  $(g_{i\bar{j}}(z, \eta))$  is positive definite, where  $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$  is the fundamental metric tensor. Equivalently, it means that the indicatrix is strongly pseudo-convex.

Consequently, from iii) we have  $\frac{\partial L}{\partial \eta^k} \eta^k = \frac{\partial L}{\partial \bar{\eta}^k} \bar{\eta}^k = L$ ,  $\frac{\partial g_{i\bar{j}}}{\partial \eta^k} \eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k} \bar{\eta}^k = 0$  and  $L = g_{i\bar{j}} \eta^i \bar{\eta}^j$ .

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold  $T'M$  endowed with the Hermitian metric structure defined by  $g_{i\bar{j}}$ .

Therefore, the first step is to study sections of the complexified tangent bundle of  $T'M$ , which is decomposed in the sum  $T_C(T'M) = T'(T'M) \oplus T''(T'M)$ . Let  $VT'M \subset T'(T'M)$  be the vertical bundle, locally spanned by  $\{\frac{\partial}{\partial \eta^k}\}$ , and  $VT''M$  be its conjugate.

At this point, the idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of this geometry. A (*c.n.c.*) is a supplementary complex subbundle to  $VT'M$  in  $T'(T'M)$ , i.e.  $T'(T'M) = HT'M \oplus VT'M$ . The horizontal distribution  $H_u T'M$  is locally spanned by  $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$ , where  $N_k^j(z, \eta)$  are the coefficients of the (*c.n.c.*), i.e. they transform by a certain rule

$$N_j^i \frac{\partial z'^j}{\partial z^k} = \frac{\partial z'^i}{\partial z^j} N_k^j - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^j. \quad (2.1)$$

The pair  $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$  will be called the adapted frame of the (c.n.c.) which obey to the change rules  $\delta_k = \frac{\partial z'^j}{\partial z^k} \delta'_j$  and  $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k} \dot{\partial}'_j$ . By conjugation everywhere we have obtained an adapted frame  $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$  on  $T''_u(T'M)$ . The dual adapted bases are  $\{dz^k, \delta\eta^k\}$  and  $\{d\bar{z}^k, \delta\bar{\eta}^k\}$ .

Let us consider  $T$  the natural tangent structure which behaves on  $T'(T'M)$  by  $T(\frac{\partial}{\partial z^k}) = \frac{\partial}{\partial \eta^k}$  ;  $T(\frac{\partial}{\partial \eta^k}) = 0$ , and it is globally defined, (see [20]).

**Definition 2.1.** [20]. A vector field  $S \in T'(T'M)$  is a complex spray if  $T \circ S = \Gamma$ , where  $\Gamma = \eta^k \frac{\partial}{\partial z^k}$  is the complex Liouville vector field.

Locally, this condition of complex spray can be expressed as follows

$$S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z, \eta) \frac{\partial}{\partial \eta^k} . \quad (2.2)$$

Under the changes of complex coordinates on  $T'M$ , the coefficients  $G^k$  of the spray  $S$  are transformed by the rule

$$2G'^i = 2G^k \frac{\partial z'^i}{\partial z^k} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^j \eta^k . \quad (2.3)$$

Between the notions of complex spray and (c.n.c.) there exists an interdependence, one determining the other. Differentiating (2.3) with respect to  $\eta^j$  it follows that the functions  $N_j^i := \frac{\partial G^i}{\partial \eta^j}$  satisfy the rule (2.1), and hence  $N_j^i$  define a nonlinear connection. Conversely, any (c.n.c.) determines a complex spray. Indeed, a simple computation shows that if  $N_j^i$  are the coefficients of a (c.n.c.) then  $\frac{1}{2} N_j^i \eta^j$  satisfy (2.3) and hence, they define a complex spray.

A (c.n.c.) related only to the fundamental function of the complex Finsler space  $(M, F)$  is the so called Chern-Finsler (c.n.c.), (cf. [1]), with the local coefficients  $N_j^i := g^{\bar{m}i} \frac{\partial g_{\bar{m}j}}{\partial z^j} \eta^l$ . Further on  $\delta_k$  is the adapted frame of the Chern-Finsler (c.n.c.). A Hermitian connection  $D$ , of  $(1, 0)$ - type, which satisfies in addition  $D_{JX}Y = JD_XY$ , for all  $X$  horizontal vectors and  $J$  the natural complex structure of the manifold, is the Chern-Finsler connection ([1]). It is locally given by the following coefficients (cf. [20]):

$$L_{jk}^i := g^{\bar{l}i} \delta_k g_{j\bar{l}} = \dot{\partial}_j N_k^i ; C_{jk}^i := g^{\bar{l}i} \dot{\partial}_k g_{j\bar{l}} . \quad (2.4)$$

Recall that  $R_{j\bar{h}k}^i := -\delta_{\bar{h}} L_{jk}^i - (\delta_{\bar{h}} N_k^l) C_{jl}^i$  are  $h\bar{h}$  - curvatures coefficients of Chern-Finsler connection. According to [1], p. 108, [20], p. 81, the holomorphic curvature of the complex Finsler space  $(M, F)$  in direction  $\eta$  is

$$\mathcal{K}_F(z, \eta) = \frac{2}{L^2} R_{\bar{r}j\bar{k}h} \bar{\eta}^r \eta^j \bar{\eta}^k \eta^h , \quad (2.5)$$

where  $R_{\bar{r}j\bar{k}h} := R_{j\bar{k}h}^i g_{i\bar{r}}$ .

In [1]'s terminology, the complex Finsler space  $(M, F)$  is *strongly Kähler* iff  $T_{jk}^i = 0$ , *Kähler* iff  $T_{jk}^i \eta^j = 0$  and *weakly Kähler* iff  $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$ , where  $T_{jk}^i := L_{jk}^i - L_{kj}^i$ . In [13] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of the complex Finsler metrics which come from Hermitian metrics on  $M$ , so-called *purely Hermitian metrics* in [20], (i.e.  $g_{i\bar{j}} = g_{i\bar{j}}(z)$ ), all those nuances of Kähler are same. On the other hand, as in Aikou's work [2], a complex Finsler space which is Kähler and  $L_{jk}^i = L_{jk}^i(z)$  is named *complex Berwald* space.

In [20] it is proved that the Chern-Finsler (*c.n.c.*) does not generally come from a complex spray except when the complex metric is weakly Kähler. But, its local coefficients  $N_j^i$  always determine a complex spray with coefficients  $G^i = \frac{1}{2} N_j^i \eta^j$ . Further,  $G^i$  induce a (*c.n.c.*) denoted by  $\overset{c}{N}_j^i := \dot{\partial}_j G^i$  and called *canonical* in [20], where it is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. With respect to the canonical (*c.n.c.*), we consider the frame  $\{\delta_k^c, \dot{\partial}_k^c\}$ , where  $\delta_k^c := \frac{\partial}{\partial z^k} - \overset{c}{N}_k^j \dot{\partial}_j$ , and its dual coframe  $\{dz^k, \overset{c}{\delta} \eta^k\}$ , where  $\overset{c}{\delta} \eta^k := d\eta^k + \overset{c}{N}_j^k dz^j$ . Moreover, we associate to the canonical (*c.n.c.*) a complex linear connection of Berwald type  $BF$  with its connection form

$$\omega_j^i(z, \eta) = G_{jk}^i dz^k + G_{j\bar{k}}^i d\bar{z}^k, \quad (2.6)$$

where  $G_{jk}^i := \dot{\partial}_k \overset{c}{N}_j^i$  and  $G_{j\bar{k}}^i := \dot{\partial}_{\bar{k}} \overset{c}{N}_j^i$ . Note that the spray coefficients perform  $2G^i = N_j^i \eta^j = \overset{c}{N}_j^i \eta^j = G_{jk}^i \eta^j \eta^k = L_{jk}^i \eta^j \eta^k$ .

An extension of the complex Berwald spaces, directly related to the  $BF$  connection, is called by us *generalized Berwald* in [3]. It is with the coefficients  $G_{jk}^i$  depending only on the position  $z$ , equivalently with either  $\dot{\partial}_h G^i = 0$  or  $BF$  is of  $(1, 0)$  - type. Since in the Kähler case  $G_{jk}^i = L_{jk}^i$ , any complex Berwald space is generalized Berwald.

In Abate-Patrizio's sense, ([1] p. 101), a complex geodesic curve is given by  $D_{T^h + \overline{T^h}} T^h = \theta^*(T^h, \overline{T^h})$ , where  $\theta^* = g^{\bar{m}k} g_{i\bar{p}} (L_{j\bar{m}}^{\bar{p}} - L_{\bar{m}j}^{\bar{p}}) dz^i \wedge d\bar{z}^j \otimes \delta_k$ , for which it is proven in [20] that  $\theta^{*k} = 2g^{\bar{j}k} \overset{c}{\delta}_{\bar{j}} L$  and  $\theta^{*i}$  is vanishing if and only if the space is weakly Kähler. Thus, the equations of a complex geodesic  $z = z(s)$  of  $(M, F)$ , with  $s$  a real parameter, in [1]'s sense can be rewritten as

$$\frac{d^2 z^i}{ds^2} + 2G^i(z(s), \frac{dz}{ds}) = \theta^{*i}(z(s), \frac{dz}{ds}) ; i = \overline{1, n}, \quad (2.7)$$

where by  $z^i(s)$ ,  $i = \overline{1, n}$ , we denote the coordinates along of curve  $z = z(s)$ .

Note that the functions  $\theta^{*i}$  are  $(1,1)$  - homogeneous with respect to  $\eta$ , i.e.  $(\dot{\partial}_k \theta^{*i})\eta^k = \theta^{*i}$  and  $(\dot{\partial}_{\bar{k}} \theta^{*i})\bar{\eta}^k = \theta^{*i}$ .

Next, we emphasize some properties of the complex Finsler spaces.

**Lemma 2.1.** *Let  $(M, F)$  be a complex Finsler space. Then,  $(\dot{\partial}_{\bar{k}} G^i)\eta_i = 0$ , where  $\eta_i := \dot{\partial}_i L$ .*

*Proof.* It results differentiating  $G^i g_{i\bar{j}} = \frac{1}{2} \frac{\partial g_{h\bar{j}}}{\partial z^s} \eta^h \eta^s$  with respect to  $\bar{\eta}^k$  and then contracting on it by  $\bar{\eta}^j$ .  $\square$

Also, it is necessary to compute

$$\begin{aligned} \dot{\partial}_k \theta^{*i} &= 2\dot{\partial}_k (g^{\bar{j}i} \delta_{\bar{j}}^c L) = -2g^{\bar{j}l} g^{\bar{m}i} (\dot{\partial}_k g_{l\bar{m}}) (\delta_{\bar{j}}^c L) + 2g^{\bar{j}i} \dot{\partial}_k (\delta_{\bar{j}}^c L) \\ &= -\theta^{*l} C_{kl}^i + 2g^{\bar{j}i} \dot{\partial}_k \left[ \frac{\partial L}{\partial \bar{z}^j} - N_{\bar{j}}^{\bar{r}} (\dot{\partial}_{\bar{r}} L) \right] \\ &= -\theta^{*l} C_{kl}^i + 2g^{\bar{j}i} \left[ \frac{\partial^2 L}{\partial \eta^k \partial \bar{z}^j} - (\dot{\partial}_k N_{\bar{j}}^{\bar{r}}) (\dot{\partial}_{\bar{r}} L) - N_{\bar{j}}^{\bar{r}} g_{k\bar{r}} \right]. \end{aligned}$$

Now, using Lemma 2.1 and  $\frac{\partial^2 L}{\partial \eta^k \partial \bar{z}^j} = N_{\bar{j}}^{\bar{r}} g_{k\bar{r}}$  we obtain

$$\begin{aligned} (\dot{\partial}_k N_{\bar{j}}^{\bar{r}}) (\dot{\partial}_{\bar{r}} L) &= (\dot{\partial}_k N_{\bar{j}}^{\bar{r}}) \bar{\eta}_r = [\dot{\partial}_{\bar{j}} (\dot{\partial}_k G^{\bar{r}})] \bar{\eta}_r = \dot{\partial}_{\bar{j}} [(\dot{\partial}_k G^{\bar{r}}) \bar{\eta}_r] - (\dot{\partial}_k G^{\bar{r}}) (\dot{\partial}_{\bar{j}} \bar{\eta}_r) \\ &= -(\dot{\partial}_k G^{\bar{r}}) C_{l\bar{r}\bar{j}} \eta^l, \text{ where } \bar{\eta}_r := \dot{\partial}_{\bar{r}} L \text{ and } C_{l\bar{r}\bar{j}} \eta^l := \dot{\partial}_{\bar{j}} \bar{\eta}_r. \end{aligned}$$

Therefore,

$$\dot{\partial}_k \theta^{*i} = -\theta^{*l} C_{kl}^i + 2g^{\bar{j}i} [(N_{\bar{j}}^{\bar{r}} - N_{\bar{j}}^{\bar{r}c}) g_{k\bar{r}} + (\dot{\partial}_k G^{\bar{r}}) C_{l\bar{r}\bar{j}} \eta^l]. \quad (2.8)$$

**Theorem 2.1.** *Let  $(M, F)$  be a complex Finsler space which is weakly Kähler and generalized Berwald. Then it is a complex Berwald space.*

*Proof.* Under given assumptions, the relation (2.8) is  $2g^{\bar{j}i} (N_{\bar{j}}^{\bar{r}} - N_{\bar{j}}^{\bar{r}c}) g_{k\bar{r}} = 0$ , which contracted by  $\frac{1}{2} g_{i\bar{m}} g^{\bar{s}k}$  gives  $N_{\bar{m}}^{\bar{s}} - N_{\bar{m}}^{\bar{s}c} = 0$ , i.e.  $F$  is Kähler. This, together with the statement of generalized Berwald, proves our claim.  $\square$

### 3 Curvature forms and Bianchi identities

We shall use the complex linear connection of Berwald type  $B\Gamma$  as our main tool to study the projective geometry of the complex Finsler manifolds. The connection form of  $B\Gamma$  satisfy the following structure equations

$$d(dz^i) - dz^k \wedge \omega_k^i = h\Omega^i; \quad d(\delta \eta^i) - \delta \eta^k \wedge \omega_k^i = v\Omega^i; \quad d\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i, \quad (3.1)$$

and their conjugates, where  $d$  is exterior differential with respect to the canonical (c.n.c.).

Since

$$\begin{aligned} d(\overset{c}{\delta} \eta^i) &= d \overset{c}{N}_j^i \wedge dz^j = \frac{1}{2} K_{jk}^i dz^k \wedge dz^j + \Theta_{j\bar{k}}^i d\bar{z}^k \wedge dz^j \\ &\quad + G_{jk}^i \overset{c}{\delta} \eta^k \wedge dz^j + G_{j\bar{k}}^i \overset{c}{\delta} \bar{\eta}^k \wedge dz^j \end{aligned}$$

and  $G_{jk}^i = G_{kj}^i$ , the torsion and curvature forms are

$$\begin{aligned} h\Omega^i &= -G_{j\bar{k}}^i dz^j \wedge d\bar{z}^k; \\ v\Omega^i &= -\frac{1}{2} K_{jk}^i dz^j \wedge dz^k - \Theta_{j\bar{k}}^i dz^j \wedge d\bar{z}^k - G_{j\bar{k}}^i dz^j \wedge \overset{c}{\delta} \bar{\eta}^k - G_{j\bar{k}}^i \overset{c}{\delta} \eta^j \wedge d\bar{z}^k; \\ \Omega_j^i &= -\frac{1}{2} K_{jkh}^i dz^k \wedge dz^h - \frac{1}{2} K_{j\bar{k}\bar{h}}^i d\bar{z}^k \wedge d\bar{z}^h + K_{j\bar{h}k}^i dz^k \wedge d\bar{z}^h \\ &\quad - G_{jkh}^i dz^k \wedge \overset{c}{\delta} \eta^h - G_{j\bar{k}\bar{h}}^i d\bar{z}^k \wedge \overset{c}{\delta} \bar{\eta}^h - G_{j\bar{h}k}^i dz^k \wedge \overset{c}{\delta} \bar{\eta}^h + G_{j\bar{h}k}^i \overset{c}{\delta} \eta^k \wedge d\bar{z}^h, \end{aligned}$$

where

$$\begin{aligned} K_{jk}^i &:= \overset{c}{\delta}_k \overset{c}{N}_j^i - \overset{c}{\delta}_j \overset{c}{N}_k^i; \quad \Theta_{j\bar{k}}^i := \overset{c}{\delta}_{\bar{k}} \overset{c}{N}_j^i; \text{ and} \\ K_{jkh}^i &:= \overset{c}{\delta}_h \overset{c}{G}_{jk}^i - \overset{c}{\delta}_k \overset{c}{G}_{jh}^i + G_{jk}^l G_{lh}^i - G_{jh}^l G_{lk}^i; \\ K_{j\bar{k}\bar{h}}^i &:= \overset{c}{\delta}_{\bar{h}} \overset{c}{G}_{j\bar{k}}^i - \overset{c}{\delta}_{\bar{k}} \overset{c}{G}_{j\bar{h}}^i + G_{j\bar{k}}^l G_{l\bar{h}}^i - G_{j\bar{h}}^l G_{l\bar{k}}^i; \\ K_{j\bar{h}k}^i &:= \overset{c}{\delta}_h \overset{c}{G}_{j\bar{k}}^i - \overset{c}{\delta}_{\bar{k}} \overset{c}{G}_{jh}^i + G_{j\bar{k}}^l G_{lh}^i - G_{jh}^l G_{l\bar{k}}^i \text{ are } hh\text{-, } \bar{h}\bar{h}\text{- and } h\bar{h}\text{- curvature} \\ &\text{tensors, respectively;} \\ G_{jkh}^i &:= \dot{\partial}_h G_{jk}^i; \quad G_{j\bar{k}\bar{h}}^i = \dot{\partial}_{\bar{h}} G_{j\bar{k}}^i; \quad G_{j\bar{h}k}^i := \dot{\partial}_h G_{j\bar{k}}^i \text{ are } hv\text{-, } \bar{h}\bar{v}\text{- and } h\bar{v}\text{-} \\ &\text{curvature tensors, respectively. Moreover, they have properties} \end{aligned}$$

$$\begin{aligned} K_{jkh}^i &= \dot{\partial}_j K_{kh}^i; \quad K_{jkh}^i \eta^j = K_{kh}^i; \quad K_{j\bar{k}\bar{h}}^i + K_{j\bar{h}\bar{k}}^i = 0; \\ G_{j\bar{h}k}^i \eta^j &= G_{h\bar{k}}^i; \quad G_{j\bar{k}\bar{h}}^i \bar{\eta}^h = -G_{j\bar{k}}^i. \end{aligned}$$

Note that we preferred to denote by  $K_{jkh}^i$  the horizontal curvature tensors of  $B\Gamma$ , instead of classical real notation  $R_{jkh}^i$ . In this way, we avoid any confusion with the horizontal curvatures coefficients of the Chern-Finsler connection from (2.5).

Taking the exterior differential of the third structure equation from (3.1), it results

$$-\Omega_j^l \wedge \omega_l^i + \omega_j^l \wedge \Omega_l^i = d\Omega_j^i, \quad (3.2)$$

which leads to sixteen Bianchi identities. We mention here only some of these, which are needed for our proposed study

$$\begin{aligned} \dot{\partial}_r G_{jkh}^i &= \dot{\partial}_h G_{jkr}^i; \quad \dot{\partial}_r G_{j\bar{h}k}^i = \dot{\partial}_h G_{j\bar{k}r}^i; \quad \dot{\partial}_r G_{j\bar{k}\bar{h}}^i = \dot{\partial}_h G_{j\bar{h}\bar{k}}^i; \\ \dot{\partial}_{\bar{r}} G_{jkh}^i &= \dot{\partial}_h G_{j\bar{r}k}^i; \quad \dot{\partial}_r G_{j\bar{k}\bar{h}}^i = \dot{\partial}_{\bar{h}} G_{j\bar{k}r}^i; \quad \dot{\partial}_{\bar{r}} G_{j\bar{h}k}^i = \dot{\partial}_{\bar{h}} G_{j\bar{r}k}^i. \end{aligned}$$

In the generalized Berwald case the following identities are true:

$$\dot{\partial}_r K_{jkh}^i = 0 ; \dot{\partial}_{\bar{r}} K_{jkh}^i = 0 ; \dot{\partial}_r K_{j\bar{k}h}^i = 0 ; \dot{\partial}_{\bar{r}} K_{j\bar{k}h}^i = 0$$

and for complex Berwald spaces we get

$$K_{j\bar{r}k|\bar{h}}^i = K_{j\bar{h}k|\bar{r}}^i ; K_{j\bar{r}k|h}^i = K_{j\bar{r}h|k}^i, \quad (3.3)$$

where we denoted by ' $|\bar{k}$ ' the horizontal covariant derivative with respect to Chern-Finsler connection.

## 4 Projective invariants of a complex Finsler space

Let  $\tilde{L}$  be another complex Finsler metric on the underlying manifold  $M$ . Corresponding to the metric  $\tilde{L}$ , we have the spray coefficients  $\tilde{G}^i$  and the functions  $\tilde{\theta}^{*i}$ . The complex Finsler metrics  $L$  and  $\tilde{L}$  on the manifold  $M$ , are called *projectively related* if these have the same complex geodesics as point sets. This means that for any complex geodesic  $z = z(s)$  of  $(M, L)$  there is a transformation of its parameter  $s$ ,  $\tilde{s} = \tilde{s}(s)$ , with  $\frac{d\tilde{s}}{ds} > 0$ , such that  $z = z(\tilde{s}(s))$  is a geodesic of  $(M, \tilde{L})$ , and conversely.

**Theorem 4.1.** [3]. *Let  $L$  and  $\tilde{L}$  be complex Finsler metrics on the manifold  $M$ . Then  $L$  and  $\tilde{L}$  are projectively related if and only if there is a smooth function  $P$  on  $T'M$  with complex values, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i ; i = \overline{1, n}, \quad (4.1)$$

where  $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$ .

The relations (4.1) between the spray coefficients  $\tilde{G}^i$  and  $G^i$  of the projectively related complex Finsler metrics  $L$  and  $\tilde{L}$  is called *projective change*. An equivalent form of this, (see Lemma 3.2, [3]) is

$$\tilde{G}^i = G^i + S\eta^i \text{ and } \tilde{\theta}^{*i} = \theta^{*i} + Q\eta^i ; i = \overline{1, n}, \quad (4.2)$$

where  $S := (\dot{\partial}_k P)\eta^k$  is  $(1, 0)$  - homogeneous,  $Q := -2(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$  is  $(0, 1)$  - homogeneous and  $S - \frac{1}{2}Q = P$ .

Differentiating (4.2) with respect to  $\eta^j$  leads to

$$\overset{c}{\tilde{N}}_j^i = \overset{c}{N}_j^i + S_j\eta^i + S\delta_j^i \text{ and } \tilde{\theta}_j^{*i} = \theta_j^{*i} + Q_j\eta^i + Q\delta_j^i, \quad (4.3)$$



where  $S_j := \dot{\partial}_j S$ ,  $Q_j := \dot{\partial}_j Q$ ,  $\tilde{\theta}_j^{*i} := \dot{\partial}_j \tilde{\theta}^{*i}$  and  $\theta_j^{*i} := \dot{\partial}_j \theta^{*i}$ . Thus,  $S_j - \frac{1}{2}Q_j = P_j$ , with  $P_j := \dot{\partial}_j P$ .

Now, to eliminate  $S$  and  $Q$  from (4.3), we make the sum by  $i = j$ . Since  $S_i \eta^i = S$  and  $Q_i \eta^i = 0$ , (4.3) gives

$$S = \frac{1}{n+1}(\tilde{N}_i^c - N_i^c) \text{ and } Q = \frac{1}{n}(\tilde{\theta}_i^{*i} - \theta_i^{*i}). \quad (4.4)$$

So that,  $P = \frac{1}{n+1}(\tilde{N}_i^c - N_i^c) - \frac{1}{2n}(\tilde{\theta}_i^{*i} - \theta_i^{*i})$ . Substituting this in (4.1), we find that the projective change is

$$\tilde{G}^i = G^i + \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i}) + \frac{1}{n+1}(\tilde{N}_l^c - N_l^c)\eta^i - \frac{1}{2n}(\tilde{\theta}_l^{*l} - \theta_l^{*l})\eta^i, \quad i = \overline{1, n}. \quad (4.5)$$

From here it results

$$D^i := G^i - \frac{1}{n+1}N_l^c \eta^i - \frac{1}{2}(\theta^{*i} - \frac{1}{n}\theta_l^{*l}\eta^i), \quad (4.6)$$

which are the components of a projective invariant, under the projective change (4.1).

**Proposition 4.1.** *Let  $(M, F)$  be a complex Finsler space. Then,  $D^i$  are the local coefficients of a complex spray if and only if  $F$  is weakly Kähler.*

*Proof.* First,  $D^i$  satisfy the rule (2.3), forasmuch  $N_l^c \eta^i$ ,  $\theta^{*i}$  and  $\theta_l^{*l}\eta^i$  have changes all like vectors. Second,  $D^i$  are  $(2, 0)$  - homogeneous if and only if  $\theta^{*i} = \frac{1}{n}\theta_l^{*l}\eta^i$ . The last relation contracted by  $\eta_i$  gives  $0 = \theta^{*i}\eta_i = \frac{1}{n}\theta_l^{*l}L$ . Hence,  $\theta_l^{*l} = 0$  and so  $\theta^{*i} = 0$ .  $\square$

Further on, the projective change (4.1) gives rise to various projective invariants. Indeed, some successive differentiations of (4.6) with respect to  $\eta$  and  $\bar{\eta}$  give three *projective curvature invariants of Douglas type*

$$\begin{aligned} D_{jkh}^i &= G_{jkh}^i - \frac{1}{n+1}[(\dot{\partial}_h D_{jk})\eta^i + \sum_{(j,k,h)} D_{jh}\delta_k^i] \\ &\quad - \frac{1}{2}\{\theta_{jkh}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{ljk}^{*l})\eta^i + \sum_{(j,k,h)} \theta_{ljk}^{*l}\delta_k^i]\}; \\ D_{j\bar{k}\bar{h}}^i &= G_{j\bar{k}\bar{h}}^i - \frac{1}{n+1}[(\dot{\partial}_{\bar{j}} D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}}\delta_j^i] \\ &\quad - \frac{1}{2}\{\theta_{j\bar{k}\bar{h}}^{*i} - \frac{1}{n}[(\dot{\partial}_{\bar{h}} \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}\bar{h}}^{*l}\delta_j^i]\}; \\ D_{j\bar{k}h}^i &= G_{j\bar{k}h}^i - \frac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j})\eta^i + D_{\bar{k}j}\delta_h^i + D_{\bar{k}h}\delta_j^i] \\ &\quad - \frac{1}{2}\{\theta_{j\bar{k}h}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}j}^{*l}\delta_h^i + \theta_{l\bar{k}h}^{*l}\delta_j^i]\}, \end{aligned} \quad (4.7)$$

where  $D_{kh} := G_{ikh}^i$ ,  $D_{\bar{k}\bar{h}} := G_{\bar{i}\bar{k}\bar{h}}^i$  and  $D_{\bar{k}h} := G_{\bar{i}k\bar{h}}^i$  are respectively,  $h\nu$ -,  $\bar{h}\bar{\nu}$ - and  $h\bar{\nu}$ - Ricci tensors and  $\theta_{jkh}^{*i} := \dot{\partial}_h \theta_{jk}^{*i}$ ,  $\theta_{jk}^{*i} := \dot{\partial}_k \theta_j^{*i}$ ,  $\theta_{j\bar{k}h}^{*i} := \dot{\partial}_{\bar{k}} \theta_{jh}^{*i}$ ,  $\theta_{j\bar{k}\bar{h}}^{*i} := \dot{\partial}_{\bar{k}} \theta_{j\bar{h}}^{*i}$  and  $\theta_{j\bar{h}}^{*i} := \dot{\partial}_{\bar{h}} \theta_j^{*i} = \dot{\partial}_j \theta_{\bar{h}}^{*i}$ . In (4.7),  $\sum_{(j,k,h)}$  is the cyclic sum.

**Definition 4.1.** A complex Finsler space  $(M, F)$  is called complex Douglas space if the invariants (4.7) are vanishing.

**Remark 4.1.** If  $F$  is generalized Berwald, i.e.  $G_{jk}^i(z)$ , and weakly Kähler then  $G_{jkh}^i = G_{j\bar{k}\bar{h}}^i = G_{j\bar{k}h}^i = 0$  and  $D_{kh} = D_{\bar{k}\bar{h}} = D_{\bar{k}h} = \theta^{*i} = 0$ , and so the projective curvature invariants of Douglas type are vanishing. Moreover, taking into account Theorem 2.1 it results that any complex Berwald space is a complex Douglas space.

Subsequently, the key of the proofs is the strong maximum principle which gives the constancy of the holomorphic and 0 - homogenous functions.

**Lemma 4.1.** If one of  $h\nu$ -,  $\bar{h}\bar{\nu}$ - or  $h\bar{\nu}$ - Ricci tensors is vanishing then they are all vanishing.

*Proof.* Supposing  $D_{kh} = 0$ , it results  $G_{ikh}^i = 0$ , which is equivalent with  $\dot{\partial}_h G_{ik}^i = 0$ . By conjugation,  $\dot{\partial}_{\bar{h}} G_{\bar{i}\bar{k}}^i = 0$ , and so,  $G_{\bar{i}\bar{k}}^i$  are holomorphic in  $\eta$ . But,  $G_{\bar{i}\bar{k}}^i$  are 0 - homogeneous with respect to  $\eta$  and so they depend only on  $z$ , ( $G_{\bar{i}\bar{k}}^i = G_{\bar{i}\bar{k}}^i(z)$ ). Hence,  $G_{ik}^i$  depend only on  $z$  and  $\dot{\partial}_{\bar{h}} G_{ik}^i = D_{\bar{h}k} = 0$  which contracted by  $\eta^k$  give  $\dot{\partial}_{\bar{h}} N_i^i = 0$ , i.e.  $G_{i\bar{h}}^i = 0$ . So that,  $D_{\bar{k}\bar{h}} = \dot{\partial}_{\bar{h}} G_{\bar{i}\bar{k}}^i = 0$ .

If  $D_{\bar{k}\bar{h}} = 0$  then  $\dot{\partial}_{\bar{h}} G_{\bar{i}\bar{k}}^i = 0$  which contracted by  $\bar{\eta}^h$  yield  $G_{\bar{i}\bar{k}}^i = 0$ , because  $G_{\bar{i}\bar{k}\bar{h}}^i \bar{\eta}^h = -G_{\bar{i}\bar{k}}^i$ . It results  $\dot{\partial}_{\bar{k}} G_{ih}^i = 0$ , i.e.  $D_{\bar{k}h} = 0$ . Further on using the holomorphicity in  $\eta$  and 0 - homogeneity of the coefficients  $G_{ih}^i$  it results that  $G_{ih}^i$  depend on  $z$  alone. So,  $\dot{\partial}_j G_{ih}^i = 0$  which give  $D_{hj} = 0$ .

If  $D_{\bar{k}h} = 0$  then  $\dot{\partial}_{\bar{k}} G_{ih}^i = 0$  and similarly it results that  $G_{ih}^i$  depend only on  $z$  and  $G_{i\bar{k}}^i = 0$ . This implies  $D_{hj} = D_{\bar{k}h} = 0$ .  $\square$

Since  $\theta^{*i}$  are  $(1,1)$  - homogeneous with respect to  $\eta$ ,  $\theta_k^{*i} \eta^k = \theta^{*i}$  and  $\theta_{\bar{k}}^{*i} \bar{\eta}^k = \theta^{*i}$  and so,

$$\begin{aligned} \theta_{kj}^{*i} \eta^k &= 0 ; \theta_{k\bar{h}}^{*i} \eta^k = \theta_{\bar{h}}^{*i} ; \theta_{\bar{k}j}^{*i} \bar{\eta}^k = \theta_j^{*i} ; \theta_{\bar{k}\bar{h}}^{*i} \bar{\eta}^k = 0; \\ \theta_{kjr}^{*i} \eta^k &= -\theta_{jr}^{*i} ; \theta_{k\bar{h}j}^{*i} \eta^k = 0 ; \theta_{r\bar{k}j}^{*i} \bar{\eta}^k = \theta_{rj}^{*i} ; \theta_{j\bar{k}\bar{h}}^{*i} \bar{\eta}^k = 0; \\ \theta_{k\bar{h}r}^{*i} \eta^k &= \theta_{h\bar{r}}^{*i} ; (\dot{\partial}_h \theta_{kjr}^{*i}) \eta^k = -2\theta_{hjr}^{*i} ; (\dot{\partial}_r \theta_{k\bar{h}j}^{*i}) \eta^k = -\theta_{r\bar{h}j}^{*i} ; \\ (\dot{\partial}_h \theta_{r\bar{k}j}^{*i}) \bar{\eta}^k &= 0 ; (\dot{\partial}_h \theta_{r\bar{k}j}^{*i}) \bar{\eta}^k = \theta_{rjh}^{*i} ; (\dot{\partial}_{\bar{r}} \theta_{k\bar{h}j}^{*i}) \bar{\eta}^k = 0. \end{aligned} \quad (4.8)$$

**Proposition 4.2.** *Let  $(M, F)$  be a complex Finsler space. If  $D_{j\bar{k}h}^i = 0$  then  $F$  is generalized Berwald and*

$$\begin{aligned} D_{jkh}^i &= -\frac{1}{2}\{\theta_{jkh}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{ljk}^{*l})\eta^i + \sum_{(j,k,h)} \theta_{ljk}^{*l} \delta_k^i]\}; \\ D_{j\bar{k}h}^i &= -\frac{1}{2}\{\theta_{j\bar{k}h}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]\}; \\ \theta_{j\bar{k}h}^{*i} &= \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}j}^{*l} \delta_h^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]. \end{aligned} \quad (4.9)$$

*Proof.* If  $D_{j\bar{k}h}^i = 0$  then

$$\begin{aligned} G_{j\bar{k}h}^i &= \frac{1}{n+1}[(\dot{\partial}_h D_{\bar{k}j}^i)\eta^i + D_{\bar{k}j}^i \delta_h^i + D_{\bar{k}h}^i \delta_j^i] \\ &\quad + \frac{1}{2}\{\theta_{j\bar{k}h}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}j}^{*l} \delta_h^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]\} \end{aligned} \quad (4.10)$$

which will be contracted by  $\eta^j \eta^h$  and then by  $\eta_i$ .

Using  $G_{j\bar{k}h}^i \eta^j \eta^h = G_{h\bar{k}}^i \eta^h = \dot{\partial}_{\bar{k}} G^i$ ;  $(\dot{\partial}_j D_{\bar{k}h}^i) \eta^j \eta^h = 0$ ;  $D_{\bar{k}h}^i \eta^h = G_{l\bar{k}}^l$  and taking into account (4.8), after the contraction by  $\eta^j \eta^h$  of  $G_{j\bar{k}h}^i$ , we obtain

$$\dot{\partial}_{\bar{k}} G^i = \frac{2}{n+1} G_{l\bar{k}}^l \eta^i.$$

Due to Lemma 2.1, i.e.  $(\dot{\partial}_{\bar{k}} G^i) \eta_i = 0$ , the contraction of the above relation with  $\eta_i$  leads to  $G_{l\bar{k}}^l = 0$ . Its differential with respect to  $\eta^h$  gives  $G_{l\bar{k}h}^l = 0$ , i.e.  $D_{\bar{k}h}^i = 0$  which plugged into (4.10) yields

$$G_{j\bar{k}h}^i = \frac{1}{2}\{\theta_{j\bar{k}h}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}j}^{*l} \delta_h^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]\}.$$

The last relation contracted by  $\eta^j$  gives  $G_{h\bar{k}}^i = 0$ . Next, it results  $\dot{\partial}_{\bar{k}} G_{jh}^i = 0$  which means that  $G_{jh}^i$  are holomorphic functions with respect to  $\eta$ . Together with their 0 - homogeneity imply  $G_{jh}^i = G_{jh}^i(z)$ . Hence  $G_{jkh}^i = G_{j\bar{k}h}^i = 0$  and (4.9).  $\square$

**Proposition 4.3.** *Let  $(M, F)$  be a complex Finsler space. If  $D_{j\bar{k}h}^i = 0$  then  $F$  is generalized Berwald and*

$$\begin{aligned} D_{jkh}^i &= -\frac{1}{2}\{\theta_{jkh}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{ljk}^{*l})\eta^i + \sum_{(j,k,h)} \theta_{ljk}^{*l} \delta_k^i]\}; \\ \theta_{j\bar{k}h}^{*i} &= \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]; \\ D_{j\bar{k}h}^i &= -\frac{1}{2}\{\theta_{j\bar{k}h}^{*i} - \frac{1}{n}[(\dot{\partial}_h \theta_{l\bar{k}j}^{*l})\eta^i + \theta_{l\bar{k}j}^{*l} \delta_h^i + \theta_{l\bar{k}h}^{*l} \delta_j^i]\}. \end{aligned} \quad (4.11)$$

*Proof.* If  $D_{j\bar{k}\bar{h}}^i = 0$  then

$$G_{j\bar{k}\bar{h}}^i = \frac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}}\delta_j^i] + \frac{1}{2}\{\theta_{j\bar{k}\bar{h}}^{*i} - \frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta_{lj\bar{k}}^{*l})\eta^i + \theta_{l\bar{k}\bar{h}}^{*l}\delta_j^i]\}. \quad (4.12)$$

The contraction of (4.12) by  $\eta^j \bar{\eta}^h \eta_i$  and using (4.8) and  $G_{j\bar{k}\bar{h}}^i \bar{\eta}^h \eta^j = -G_{j\bar{k}}^i \eta^j = -\dot{\partial}_{\bar{k}} G^i$ ;  $D_{\bar{k}\bar{h}} \bar{\eta}^h = -G_{i\bar{k}}^i$ ;  $(\dot{\partial}_j D_{\bar{k}\bar{h}}) \bar{\eta}^h \eta^j = -(\dot{\partial}_j G_{i\bar{k}}^i) \eta^j = -G_{i\bar{k}}^i$ , it results

$$0 = -(\dot{\partial}_{\bar{k}} G^i) \eta_i = -\frac{2L}{n+1} G_{l\bar{k}}^l,$$

which implies  $G_{l\bar{k}}^l = 0$  and so,  $G_{l\bar{k}\bar{h}}^l = 0$ , i.e.  $D_{\bar{k}\bar{h}} = 0$ . Plugging  $D_{\bar{k}\bar{h}} = 0$  into (4.12) we obtain

$$G_{j\bar{k}\bar{h}}^i = \frac{1}{2}\{\theta_{j\bar{k}\bar{h}}^{*i} - \frac{1}{n}[(\dot{\partial}_{\bar{h}}\theta_{lj\bar{k}}^{*l})\eta^i + \theta_{l\bar{k}\bar{h}}^{*l}\delta_j^i]\}.$$

Now, the last relations contracted only by  $\bar{\eta}^h$  leads to  $G_{j\bar{k}}^i = 0$ . As above we obtain that  $G_{j\bar{h}}^i$  depend only on  $z$ . So, the space is generalized Berwald and the relations (4.11) are true.  $\square$

**Theorem 4.2.** *Let  $(M, F)$  be a complex Finsler space. Then,  $(M, F)$  is Douglas if and only if it is generalized Berwald with*

$$\theta_{jkh}^{*i} = \frac{1}{n}[(\dot{\partial}_h \theta_{lj\bar{k}}^{*l})\eta^i + \sum_{(j,k,h)} \theta_{lj\bar{h}}^{*l} \delta_k^i]; \quad (4.13)$$

$$\theta_{j\bar{k}\bar{h}}^{*i} = \frac{1}{n}[(\dot{\partial}_{\bar{h}} \theta_{lj\bar{k}}^{*l})\eta^i + \theta_{l\bar{k}\bar{h}}^{*l} \delta_j^i];$$

$$\theta_{j\bar{k}h}^{*i} = \frac{1}{n}[(\dot{\partial}_h \theta_{lj\bar{k}}^{*l})\eta^i + \theta_{lj\bar{k}}^{*l} \delta_h^i + \theta_{l\bar{k}h}^{*l} \delta_j^i].$$

*Proof.* The direct implication is obvious by the last two Propositions. Conversely, if the space is generalized Berwald, replacing the relations (4.13) into (4.7), it follows  $D_{j\bar{k}h}^i = D_{jkh}^i = D_{j\bar{k}\bar{h}}^i = 0$ .  $\square$

## 5 Weakly Kähler projective changes

All the next discussion will be focused on the weakly Kähler complex Finsler spaces. In this case, the projective invariants of Douglas type (4.7) are

$$D_{jkh}^i = G_{jkh}^i - \frac{1}{n+1}[(\dot{\partial}_j D_{kh})\eta^i + \sum_{(j,k,h)} D_{jk} \delta_h^i]; \quad (5.1)$$

$$D_{j\bar{k}\bar{h}}^i = G_{j\bar{k}\bar{h}}^i - \frac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}\bar{h}})\eta^i + D_{\bar{k}\bar{h}} \delta_j^i];$$

$$D_{j\bar{k}h}^i = G_{j\bar{k}h}^i - \frac{1}{n+1}[(\dot{\partial}_j D_{\bar{k}h})\eta^i + D_{\bar{k}h} \delta_j^i + D_{\bar{k}j} \delta_h^i].$$

By Lemma 4.1, it immediately results the following Proposition.

**Proposition 5.1.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space. If one of  $hv$ -,  $\bar{h}\bar{v}$ - or  $h\bar{v}$ - Ricci tensors is vanishing, then*

$$D_{jkh}^i = G_{jkh}^i ; D_{j\bar{k}\bar{h}}^i = G_{j\bar{k}\bar{h}}^i ; D_{j\bar{k}h}^i = G_{j\bar{k}h}^i. \quad (5.2)$$

**Proposition 5.2.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space. If one of statements (5.2) is true, then the  $hv$ -,  $\bar{h}\bar{v}$ - and  $h\bar{v}$ - Ricci tensors are vanishing.*

*Proof.* Suppose that  $D_{j\bar{k}\bar{h}}^i = G_{j\bar{k}\bar{h}}^i$ . Then, using (5.1) it results  $(\dot{\partial}_j D_{\bar{k}\bar{h}}) \eta^i + D_{\bar{k}\bar{h}}^i \delta_j^i = 0$ . Since  $(\dot{\partial}_j D_{\bar{k}\bar{h}}) \eta^j = D_{\bar{k}\bar{h}}$ , hence  $(n+1)D_{\bar{k}\bar{h}} = 0$ , and so  $D_{\bar{k}\bar{h}} = 0$ . By Lemma 4.1,  $hv$ -, and  $h\bar{v}$ - Ricci tensors are vanishing. The proof is similar for  $D_{jkh}^i = G_{jkh}^i$  or  $D_{j\bar{k}h}^i = G_{j\bar{k}h}^i$ .  $\square$

Corroborating (5.1) with Propositions 4.2 and 4.3, it follows

**Corollary 5.1.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space.*

- i) *If  $D_{j\bar{k}h}^i = 0$  then  $D_{jkh}^i = D_{j\bar{k}\bar{h}}^i = 0$ .*
- ii) *If  $D_{j\bar{k}\bar{h}}^i = 0$  then  $D_{jkh}^i = D_{j\bar{k}h}^i = 0$ .*

**Theorem 5.1.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space. If either  $D_{j\bar{k}\bar{h}}^i = 0$  or  $D_{j\bar{k}h}^i = 0$  then the space is complex Berwald.*

*Proof.* If either  $D_{j\bar{k}\bar{h}}^i = 0$  or  $D_{j\bar{k}h}^i = 0$  then  $G_{jh}^i = G_{jh}^i(z)$ , which means that the space is generalized Berwald. The proof is completed by Theorem 2.1.  $\square$

**Theorem 5.2.** *If  $(M, F)$  is a complex weakly Kähler Douglas space then it is Berwald.*

*Proof.* It results by Theorem 5.1.  $\square$

Note that, the weakly Kähler property is preserved by the projective changes (for proof details see Theorem 3.2 from [3]), and then we have

$$\tilde{G}^i = G^i + P\eta^i, \quad (5.3)$$

where  $P$  is a  $(1, 0)$  - homogeneous function. Under this projective change, we obtain

$$\begin{aligned} \tilde{N}_j^i &= N_j^i + P_j \eta^i + P \delta_j^i ; \tilde{\delta}_k^i = \delta_k^i - (P_k \eta^i + P \delta_k^i) \dot{\partial}_i ; \\ \tilde{G}_{jk}^i &= G_{jk}^i + P_{jk} \eta^i + P_k \delta_j^i + P_j \delta_k^i ; \tilde{G}_{j\bar{k}}^i = G_{j\bar{k}}^i + P_{j\bar{k}} \eta^i + P_{\bar{k}} \delta_j^i, \end{aligned} \quad (5.4)$$

where  $P_{jk} := \dot{\partial}_k P_j = P_{kj}$ ,  $P_{\bar{k}} := \dot{\partial}_{\bar{k}} P$ ,  $P_{j\bar{k}} := \dot{\partial}_{\bar{k}} P_j = \dot{\partial}_j P_{\bar{k}}$ . Moreover, the  $(1, 0)$  - homogeneity of  $P$  implies

$$P_k \eta^k = P ; P_{j\bar{k}} \bar{\eta}^k = 0 ; P_{jk} \eta^k = 0 ; P_{\bar{k}} \bar{\eta}^k = 0 ; P_{j\bar{k}} \eta^j = P_{\bar{k}}. \quad (5.5)$$

Next, we shall study the  $hh$ - curvatures tensor  $K_{jkh}^i$ . Under the projective change (5.3), we have

$$\begin{aligned} \tilde{K}_{kh}^i &= K_{kh}^i + \mathcal{A}_{(k,h)} [P_{k|h}^B \eta^i + (P_{B|h} - PP_h) \delta_k^i]; \\ \tilde{K}_{jkh}^i &= K_{jkh}^i + \mathcal{A}_{(k,h)} [P_{jk|h}^B \eta^i + P_{k|h}^B \delta_j^i + (P_{j|h}^B - P_j P_h - PP_{jh}) \delta_k^i], \end{aligned} \quad (5.6)$$

where  $'_{\substack{B \\ |h}}$  is the horizontal covariant derivative with respect to  $B\Gamma$  and  $\mathcal{A}_{(k,h)}$  is the alternate operator, for example  $\mathcal{A}_{(k,h)} \{P_{k|h}^B\} := P_{k|h}^B - P_{h|k}^B$ . Next we make the following notations

$$X_{kh} := P_{k|h}^B - P_{h|k}^B ; X_h := P_{|h}^B - PP_h$$

which have the properties

$$\begin{aligned} \dot{\partial}_j X_h &= P_{j|h}^B - P_j P_h - PP_{jh} ; \dot{\partial}_j X_h - \dot{\partial}_h X_j = P_{j|h}^B - P_{h|j}^B = X_{jh} ; \\ \dot{\partial}_j X_{kh} &= P_{kj|h}^B - P_{hj|k}^B ; (\dot{\partial}_j X_h) \eta^j = X_h ; (\dot{\partial}_j X_{kh}) \eta^j = 0 ; \\ X_{kj} \eta^j &= P_{k|0}^B - P_{|k}^B := X_{k0}. \end{aligned}$$

By means of these, the changes (5.6) become

$$\begin{aligned} \tilde{K}_{kh}^i &= K_{kh}^i + X_{kh} \eta^i + X_h \delta_k^i - X_k \delta_h^i \\ \tilde{K}_{jkh}^i &= K_{jkh}^i + (\dot{\partial}_j X_{kh}) \eta^i + X_{kh} \delta_j^i + (\dot{\partial}_j X_h) \delta_k^i - (\dot{\partial}_j X_k) \delta_h^i. \end{aligned} \quad (5.7)$$

Now, we introduce the  $hh$ - Ricci tensor  $K_{kh} := K_{ikh}^i$ . Another important tensor is  $H_{jk} := K_{jki}^i$ . The link between these horizontal curvature tensors is  $H_{kj} - H_{jk} = K_{jk}^i$ . Summing by  $i = j$  and then  $i = h$  together with a contraction by  $\eta^j$ , in the second relation from (5.7), it yields

$$\begin{aligned} X_{kh} &= \frac{1}{n+1} (\tilde{K}_{kh} - K_{kh}) = \frac{1}{n+1} [(\tilde{H}_{hk} - \tilde{H}_{kh}) - (H_{hk} - H_{kh})]; \\ \tilde{H}_{0k} &= H_{0k} + X_{k0} - (n-1)X_k. \end{aligned} \quad (5.8)$$

From here, it results

$$\begin{aligned} X_{k0} &= \frac{1}{n+1} [(\tilde{H}_{0k} - \tilde{H}_{k0}) - (H_{0k} - H_{k0})]; \\ X_k &= -\frac{1}{n+1} (\tilde{H}_k - H_k) \text{ with } H_k := \frac{1}{n-1} (nH_{0k} + H_{k0}), \end{aligned} \quad (5.9)$$

for any  $n \geq 2$ . Moreover,

$$K_{jk} = \dot{\partial}_j H_{k0} - \dot{\partial}_k H_{j0} = \dot{\partial}_k H_{0j} - \dot{\partial}_j H_{0k} \text{ and } H_{jk} = \dot{\partial}_j H_{0k}. \quad (5.10)$$

Now, substituting (5.8) and (5.9) in (5.7) we obtain the following invariants

$$\begin{aligned} W_{kh}^i &= K_{kh}^i + \frac{1}{n+1} \mathcal{A}_{(k,h)}(H_{kh}\eta^i + H_h\delta_k^i); \\ W_{jkh}^i &= K_{jkh}^i + \frac{1}{n+1} \mathcal{A}_{(k,h)}[(\dot{\partial}_j H_{kh})\eta^i + H_{kh}\delta_j^i + (\dot{\partial}_j H_h)\delta_k^i], \end{aligned} \quad (5.11)$$

in which the second formula is a *projective curvature invariant of Weyl type*. Note that, if  $(M, F)$  is Kähler, then  $W_{jkh}^i = 0$ .

**Theorem 5.3.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space of complex dimension  $n \geq 2$ .*

- i) *Then,  $W_{jkh}^i = 0$  if and only if  $W_{kh}^i = 0$ ;*
- ii) *If  $K_{kh} = 0$  then  $W_{jkh}^i = K_{jkh}^i + \frac{1}{n-1}(H_{jh}\delta_k^i - H_{jk}\delta_h^i)$ ;*
- iii) *If  $H_{kh} = 0$  then  $W_{jkh}^i = K_{jkh}^i$ .*

*Proof.* i) If  $W_{jkh}^i = 0$ , then

$$K_{jkh}^i = -\frac{1}{n+1} \mathcal{A}_{(k,h)}[(\dot{\partial}_j H_{kh})\eta^i + H_{kh}\delta_j^i + (\dot{\partial}_j H_h)\delta_k^i],$$

which contracted by  $\eta^j$  give  $K_{kh}^i = -\frac{1}{n+1} \mathcal{A}_{(k,h)}(H_{kh}\eta^i + H_h\delta_k^i)$  and hence,  $W_{kh}^i = 0$ .

Conversely, if  $W_{kh}^i = 0$  then  $K_{kh}^i = -\frac{1}{n+1} \mathcal{A}_{(k,h)}(H_{kh}\eta^i + H_h\delta_k^i)$ . Differentiating with respect to  $\eta^j$ , it results

$$K_{jkh}^i = -\frac{1}{n+1} \mathcal{A}_{(k,h)}[(\dot{\partial}_j H_{kh})\eta^i + H_{kh}\delta_j^i + (\dot{\partial}_j H_h)\delta_k^i], \text{ that is, } W_{jkh}^i = 0.$$

ii) If  $K_{kh} = 0$  then  $H_{kj} = H_{jk}$ . Substituting into (5.11) and using (5.9) and (5.10), it results our claim. iii) immediately results by (5.11) and (5.9).  $\square$

In order to obtain another projective curvature invariant of Weyl type we assume that the weakly Kähler complex Finsler metric  $F$  is generalized Berwald. Thus, we have  $K_{j\bar{k}h}^i = 0$ ,  $K_{j\bar{k}h}^i = -\delta_{\bar{k}}^c G_{jh}^i$  and the Bianchi identities get  $\partial_r K_{j\bar{k}h}^i = 0$  and  $\partial_{\bar{r}} K_{j\bar{k}h}^i = 0$ .

Note that by a projective change, the generalized Berwald property of the metric  $L$  is transferred to the metric  $\tilde{L}$ . Moreover, the generalized Berwald property together with the weakly Kähler assumption implies that  $F$  and  $\tilde{F}$  are complex Berwald metrics (Theorem 2.1). Hence,  $K_{j\bar{k}h}^i = -\delta_{\bar{k}}^c L_{jh}^i$ . Therefore, under these assumptions, the function  $P$  from the projective change (5.3) is holomorphic with respect to  $\eta$ , i.e.  $P_{\bar{k}} = 0$ , (see Proposition 3.1 from [3]), and

$$\begin{aligned} \tilde{N}_j^i &= N_j^i + P_j\eta^i + P\delta_j^i; \quad \tilde{\delta}_k = \delta_k - (P_k\eta^i + P\delta_k^i)\dot{\partial}_i; \\ \tilde{L}_{jk}^i &= L_{jk}^i + P_{jk}\eta^i + P_k\delta_j^i + P_j\delta_k^i; \quad \tilde{G}_{j\bar{k}}^i = G_{j\bar{k}}^i = 0. \end{aligned} \quad (5.12)$$

Consequently,

$$\begin{aligned}\tilde{K}_{j\bar{k}h}^i &= K_{j\bar{k}h}^i - P_{jh|\bar{k}}\eta^i - P_{j|\bar{k}}\delta_h^i - P_{h|\bar{k}}\delta_j^i; \\ 0 &= P_{jhr|\bar{k}}\eta^i + P_{jh|\bar{k}}\delta_r^i + P_{jr|\bar{k}}\delta_h^i + P_{hr|\bar{k}}\delta_j^i.\end{aligned}\quad (5.13)$$

Next, we consider the  $h\bar{h}$  - Ricci tensor  $K_{\bar{k}h} := K_{i\bar{k}h}^i$ . Since  $F$  is Kähler,  $K_{i\bar{k}h}^i = K_{h\bar{k}i}^i$ . Making  $i = j$  in (5.13), it results

$$\begin{aligned}P_{h|\bar{k}} &= -\frac{1}{n+1}(\tilde{K}_{\bar{k}h} - K_{\bar{k}h}); \\ P_{hr|\bar{k}} &= 0,\end{aligned}\quad (5.14)$$

which substituted into the first equation from (5.13), give a new *projective curvature invariant of Weyl type*, which is valid only for the complex Berwald spaces,

$$W_{j\bar{k}h}^i = K_{j\bar{k}h}^i - \frac{1}{n+1}(K_{\bar{k}j}\delta_h^i + K_{\bar{k}h}\delta_j^i). \quad (5.15)$$

Note that for any complex Berwald space, the  $h\bar{h}$  - curvatures coefficients of Chern-Finsler connection can be rewritten as  $R_{j\bar{k}h}^i = K_{j\bar{k}h}^i + K_{m\bar{k}h}^l \eta^m C_{jl}^i$ . So that,  $R_{\bar{r}j\bar{k}h} = K_{\bar{r}j\bar{k}h} + K_{m\bar{k}h}^l \eta^m C_{j\bar{r}l}$ , where  $K_{\bar{r}j\bar{k}h} := K_{j\bar{k}h}^i g_{i\bar{r}}$ , and  $R_{\bar{r}j\bar{k}h} \eta^j = K_{\bar{r}j\bar{k}h} \eta^j$ . This implies

$$\mathcal{K}_F(z, \eta) = \frac{2}{L^2} K_{\bar{r}j\bar{k}h} \bar{\eta}^r \eta^j \bar{\eta}^k \eta^h.$$

**Theorem 5.4.** *Let  $(M, F)$  be a connected complex Berwald space of complex dimension  $n \geq 2$ . Then,  $W_{j\bar{k}h}^i = 0$  if and only if  $K_{\bar{m}j\bar{k}h} = \frac{\mathcal{K}_F}{4}(g_{j\bar{k}}g_{h\bar{m}} + g_{h\bar{k}}g_{j\bar{m}})$ . In this case,  $\mathcal{K}_F = c$ , where  $c$  is a constant on  $M$  and the space is either purely Hermitian with  $K_{\bar{k}j} = \frac{c(n+1)}{4}g_{j\bar{k}}$  or non purely Hermitian with  $c = 0$  and  $K_{j\bar{k}h}^i = 0$ .*

*Proof.* Using (5.15) and  $W_{j\bar{k}h}^i = 0$ , it results

$$K_{j\bar{k}h}^i = \frac{1}{n+1}(K_{\bar{k}j}\delta_h^i + K_{\bar{k}h}\delta_j^i) \quad (5.16)$$

which contracted with  $g_{i\bar{m}}$  gives

$$K_{\bar{m}j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}g_{h\bar{m}} + K_{\bar{k}h}g_{j\bar{m}}), \quad (5.17)$$

and

$$R_{\bar{m}j\bar{k}h} = \frac{1}{n+1}(K_{\bar{k}j}g_{h\bar{m}} + K_{\bar{k}h}g_{j\bar{m}} + K_{\bar{k}l}\eta^l C_{j\bar{m}h}), \quad (5.18)$$



where  $C_{j\bar{m}h} := \dot{\partial}_h g_{j\bar{m}}$ .

Since  $R_{\bar{r}j\bar{k}h} = R_{\bar{r}h\bar{k}j}$ , see [1] p. 105, it results  $R_{\bar{r}j\bar{k}h} = R_{\bar{k}j\bar{r}h}$ , and therefore,

$$K_{\bar{r}j\bar{k}h}\eta^j = K_{\bar{k}j\bar{r}h}\eta^j. \quad (5.19)$$

From (5.17) also results

$$\mathcal{K}_F = \frac{4}{L(n+1)} K_{\bar{k}j}\eta^j\bar{\eta}^k, \quad (5.20)$$

which, indeed, can be rewritten as  $L\mathcal{K}_F = \frac{4}{n+1} K_{\bar{k}j}\eta^j\bar{\eta}^k$ . Differentiating this last formula with respect to  $\bar{\eta}^m$  and using again the Bianchi identity  $\dot{\partial}_{\bar{m}} K_{\bar{k}h} = 0$ , it follows that  $\mathcal{K}_F\bar{\eta}_m + L(\dot{\partial}_{\bar{m}}\mathcal{K}_F) = \frac{4}{n+1} K_{\bar{m}j}\eta^j$ .

Now, due to (5.19), we obtain

$$\mathcal{K}_F\bar{\eta}_m = \frac{4}{n+1} K_{\bar{m}j}\eta^j. \quad (5.21)$$

Thus,  $L(\dot{\partial}_{\bar{m}}\mathcal{K}_F) = 0$  and so,  $\mathcal{K}_F$  depends only on  $z$ . Differentiating (5.21) with respect to  $\eta^l$ , it gives  $K_{\bar{m}l} = \frac{(n+1)\mathcal{K}_F}{4} g_{l\bar{m}}$ , which plugged into (5.17) yields  $K_{\bar{m}j\bar{k}h} = \frac{\mathcal{K}_F}{4} (g_{j\bar{k}}g_{h\bar{m}} + g_{h\bar{k}}g_{j\bar{m}})$ .

Conversely, since  $K_{j\bar{k}h}^i = \frac{\mathcal{K}_F}{4} (g_{j\bar{k}}\delta_h^i + g_{h\bar{k}}\delta_j^i)$  and  $K_{\bar{k}h} = \frac{(n+1)\mathcal{K}_F}{4} g_{h\bar{k}}$ , the relation (5.15) implies  $W_{j\bar{k}h}^i = 0$ .

In order to prove that  $\mathcal{K}_F$  is a constant on  $M$  we use the Bianchi identity  $K_{j\bar{r}k|\bar{h}}^i = K_{j\bar{h}k|\bar{r}}^i$  from (3.3). Contracting by  $g_{i\bar{m}}\bar{\eta}^m\eta^j\bar{\eta}^r\eta^k$ , it gives

$$\mathcal{K}_{F|\bar{h}} = \frac{1}{L} \mathcal{K}_{F|\bar{0}}\bar{\eta}_h. \quad (5.22)$$

Taking into account  $\mathcal{K}_{F|\bar{h}}|_j = \mathcal{K}_F|_{j|\bar{h}} = 0$ , where  $|_k$  is the vertical covariant derivative with respect to Chern-Finsler connection, and deriving (5.22), we easily deduce

$$0 = \mathcal{K}_{F|\bar{h}}|_j = \frac{1}{L} \mathcal{K}_{F|\bar{0}}(g_{j\bar{h}} - \frac{1}{L} \eta_j\bar{\eta}_h),$$

which multiplied by  $g^{\bar{h}j}$ , it gets  $\frac{1}{L}(n-1)\mathcal{K}_{F|\bar{0}} = 0$ . Plugging it into (5.22), it follows that  $\mathcal{K}_{F|\bar{h}} = 0$ , i.e.  $\frac{\partial \mathcal{K}_F}{\partial \bar{z}^h} = 0$ . By conjugation,  $\frac{\partial \mathcal{K}_F}{\partial z^h} = 0$  and so,  $\mathcal{K}_F$  is a constant  $c$  on  $M$ . This implies  $K_{\bar{k}j} = \frac{c(n+1)}{4} g_{j\bar{k}}$  and its derivative with respect to  $\eta^l$  leads to  $c \dot{\partial}_l g_{j\bar{k}} = 0$ , and hence the last claim.  $\square$

## 6 Locally projectively flat complex Finsler metrics

Using some ideas from the real case, we shall define the locally projectively flat complex Finsler metrics.

Let  $\tilde{L}$  be a locally Minkowski complex Finsler metric on the underlying manifold  $M$ . Corresponding to the metric  $\tilde{L}$ , at any point of  $M$  there exist local charts in which the fundamental metric tensor  $\tilde{g}_{i\bar{j}}$  depends only on  $\eta$  and thus, the spray coefficients  $\tilde{G}^i = 0$  and the functions  $\tilde{\theta}^{*i} = 0$ , in such local charts. The complex Finsler metrics  $L$  will be called *locally projectively flat* if it is projectively related to the locally Minkowski metric  $\tilde{L}$ . Since the weakly Kähler property is preserved under the projective change, any locally projectively flat metric is weakly Kähler. We recall Theorem 3.3 from [3],

**Theorem 6.1.** *Let  $L$  and  $\tilde{L}$  be complex Finsler metrics on the manifold  $M$ . Then,  $L$  and  $\tilde{L}$  are projectively related if and only if*

$$\frac{1}{2}[\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L})] = P(\dot{\partial}_{\bar{r}}\tilde{L}) + B^i \tilde{g}_{i\bar{r}} ; r = \overline{1, n}, \quad (6.1)$$

with  $P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})]$  and  $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$ .

**Theorem 6.2.**  *$L$  is locally projectively flat if and only if it is weakly Kähler and*

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(\dot{\partial}_l \tilde{L}) = 2P(\dot{\partial}_{\bar{r}}\tilde{L}) ; r = \overline{1, n}, \quad (6.2)$$

where  $P = \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k$ . Moreover,  $G^i = -P\eta^i$ .

*Proof.* The above equivalence results by Theorem 6.1 in which  $\tilde{L}$  is a locally Minkowski metric on  $M$ . Taking into account  $(\delta_k \tilde{L})\eta^k = -2G^l(\dot{\partial}_l \tilde{L})$ , the condition (6.2) is equivalent to  $-G^l \tilde{g}_{l\bar{r}} = P(\dot{\partial}_{\bar{r}}\tilde{L})$ . By contraction with  $\tilde{g}^{\bar{r}i}$ , we obtain  $G^i = -P\eta^i$ .  $\square$

**Proposition 6.1.** *If  $L$  is locally projectively flat then  $G^i = \frac{1}{2L} \frac{\partial L}{\partial z^k} \eta^k \eta^i$ .*

*Proof.* Since  $G^i = \frac{1}{2}g^{\bar{m}i} \frac{\partial g_{r\bar{m}}}{\partial z^k} \eta^k \eta^r$  and  $L$  is locally projectively flat, then  $\frac{1}{2}g^{\bar{m}i} \frac{\partial g_{r\bar{m}}}{\partial z^k} \eta^k \eta^r = -P\eta^i$ . Contracting by  $\eta_i$ , it leads to  $P = -\frac{1}{2L} \frac{\partial L}{\partial z^k} \eta^k$  which finishes the proof.  $\square$

**Proposition 6.2.** *Let  $(M, F)$  be a generalized Berwald space. If  $L$  is locally projectively flat then it is a complex Berwald metric with  $W_{j\bar{k}h}^i = 0$ .*

*Proof.* By Theorem 2.1,  $L$  is a complex Berwald metric. Since  $\tilde{K}_{j\bar{k}h}^i = \tilde{K}_{\bar{k}h}^i = 0$ , the relations (5.13) and (5.14), give  $K_{j\bar{k}h}^i = \frac{1}{n+1}(K_{\bar{k}j}\delta_h^i + K_{\bar{k}h}\delta_j^i)$  and so,  $W_{j\bar{k}h}^i = 0$ .  $\square$

By Theorem 5.4 we have proved

**Theorem 6.3.** *Let  $(M, F)$  be a connected generalized Berwald space of complex dimension  $n \geq 2$ . If  $L$  is locally projectively flat then it is of constant holomorphic curvature. Moreover, if the constant value of the holomorphic curvature is non-zero, then  $(M, F)$  is a purely Hermitian space.*

Next we study as an application the weakly Kähler complex Finsler metrics  $L$  with the spray coefficients  $G^i = \rho_r \eta^r \eta^i$ , where  $\rho$  is a smooth complex function depending only on  $z \in M$ ,  $\rho_r := \frac{\partial \rho}{\partial z^r}$  and  $\rho_{r\bar{h}} := \frac{\partial \rho_r}{\partial \bar{z}^h}$  is Hermitian, i.e.  $\overline{\rho_{r\bar{h}}} = \rho_{h\bar{r}}$ , and it is nondegenerated.

**Theorem 6.4.** *Let  $(M, F)$  be a weakly Kähler complex Finsler space with  $G^i = \rho_r \eta^r \eta^i$ . Then*

- i)  $L$  is locally projectively flat;
- ii)  $L$  is a complex Berwald metric;
- iii)  $L$  is a purely Hermitian metric of non-zero constant holomorphic curvature  $\mathcal{K}_F = -\frac{4}{L}\rho_{r\bar{h}}\eta^r\bar{\eta}^h$ .
- iv)  $\rho$  satisfies the system of partial differential equations

$$\rho_{r\bar{h}k} = \rho_r \rho_{k\bar{h}} + \rho_k \rho_{r\bar{h}}, \quad (6.3)$$

where  $\rho_{r\bar{h}k} := \frac{\partial \rho_{r\bar{h}}}{\partial z^k} = \frac{\partial \rho_{k\bar{h}}}{\partial z^r} = \frac{\partial \rho_{rk}}{\partial \bar{z}^h}$  and  $\rho_{r\bar{h}k} = \rho_{k\bar{h}r}$

*Proof.* In order to prove i), we use Theorem 6.2. Let  $\tilde{L}$  be a locally Minkowski metric on  $M$ . Since  $L$  is weakly Kähler, we must show only that the equation (6.2) is satisfied. Indeed, we have  $\dot{\partial}_{\bar{r}} G^l = 0$ ,  $(\delta_k \tilde{L})\eta^k = -2G^l(\dot{\partial}_l \tilde{L}) = -2\rho_r \eta^r \eta^l(\dot{\partial}_l \tilde{L}) = -2\tilde{L}\rho_r \eta^r$ , and so  $\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k = -2(\dot{\partial}_{\bar{r}} \tilde{L})\rho_l \eta^l$ , which implies the equation (6.2).

Since  $\dot{\partial}_{\bar{r}} G^l = 0$ ,  $L$  is generalized Berwald. Thus, Theorem 2.1 yields ii).

iii) Theorem 6.3 together with i) and ii) show that  $W_{j\bar{k}h}^i = 0$  and  $L$  is of constant holomorphic curvature. Since  $L$  is a complex Berwald metric,  $\delta_{\bar{k}} = \delta_{\bar{k}}^c$  and  $L_{jh}^i = G_{jh}^i$ . Hence  $K_{j\bar{k}h}^i = -\delta_{\bar{k}}^c L_{jh}^i$ , which will be rewritten in terms of derivatives of  $\rho$ . Indeed, two successive differentiations of the equations  $G^i = \rho_r \eta^r \eta^i$  lead to

$$L_{jk}^i = \rho_k \delta_j^i + \rho_j \delta_k^i. \quad (6.4)$$

Consequently,

$$K_{j\bar{k}h}^i = -\rho_{j\bar{k}} \delta_h^i - \rho_{h\bar{k}} \delta_j^i$$

which gives  $K_{\bar{r}j\bar{k}h} = -\rho_{j\bar{k}}g_{h\bar{r}} - \rho_{h\bar{k}}g_{j\bar{r}}$  and so,

$$\mathcal{K}_F = -\frac{4}{L}\rho_{r\bar{h}}\eta^r\bar{\eta}^h. \quad (6.5)$$

Since  $\rho_{r\bar{h}}$  is nondegenerated,  $\mathcal{K}_F \neq 0$  and by Theorem 6.3 it results that  $L$  is a purely Hermitian metric.

iv) To establish the system (6.3) we use (6.5). This implies

$$L = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}}\eta^r\bar{\eta}^h = g_{r\bar{h}}\eta^r\bar{\eta}^h, \quad (6.6)$$

which gives

$$g_{r\bar{h}} = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}} \text{ and } \delta_k g_{r\bar{h}} = -\frac{4}{\mathcal{K}_F}\rho_{r\bar{h}k}. \quad (6.7)$$

Now, using (2.4) and (6.4) it results

$$\delta_k g_{j\bar{m}} = \rho_k g_{j\bar{m}} + \rho_j g_{k\bar{m}} \quad (6.8)$$

The substitution of (6.7) into (6.8) implies (6.3). Moreover, the Kähler property of  $L$  gives  $\rho_{r\bar{h}k} = \rho_{k\bar{h}r}$ .  $\square$

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